Fendley, Ludwig and Saleur reply to Skorik's comment:

In a series of papers [1], we have used a novel approach, combining the Bethe ansatz with a kinetic (Boltzmann) equation, in order to compute exactly transport properties in a Luttinger liquid with an impurity (also known as the boundary sine-Gordon model [BSGM]). In a recent comment [2], Skorik claimed that while our results were correct in linear response, there was a "serious flaw" in our calculation at finite voltage. We explain here why Skorik's arguments are inappropriate, and that there is no flaw in our work.

In our approach, the effect of the voltage corresponds to injecting into the sample left movers and right movers from different reservoirs, at chemical potentials, $\mu_{L,R} = \pm\frac{eV}{2}$ respectively [3,4]. We use a Landauer-Büttiker scattering approach [5]. By a non-linear change of basis called "folding" [1], we map this to a problem with reservoirs setting different chemical potentials for the quasiparticles of **charge** $\pm$. These quasiparticles are [6] the "massless limit" of the usual sine-Gordon soliton and antisoliton states. Because the model is integrable, these scatter one-by-one off each other and off the impurity, conserving their momentum and either conserving or switching charge, with exactly known reflection and transmission S-matrices. This permits the construction of the exact (scattering) eigenstates of the *interacting* Hamiltonian, describing the Luttinger liquid leads *plus* the impurity, in a way analogous to ordinary potential scattering. To compute the conductance, we use a Boltzmann equation, counting how much charge is transported through the impurity in the presence of different populations of $\pm$ quasiparticles, set by the bias. This is natural, in spite of the interacting nature of the problem, because of the very simple nature of collisions that follows from integrability. In fact, we note that subsequent to our original work, the Boltzmann equation, in the case of linear response, was derived [7] directly from the Kubo formula combined with form-factors, i.e. matrix elements of current operators in the quasi particle basis, and complete agreement with the earlier results [1] was found.

The use of the Boltzmann equation **per se** was not criticized by Skorik. As far as we understand it, his concern is that in the Boltzmann equation we used scattering matrix elements "at zero voltage", whereas, for finite voltage, the S matrix itself might potentially acquire a voltage dependence, determined by the V dependent filling of the ground state. We will show that this is in fact *not* the case. First, we note that following Skorik's logic, the same criticism could seemingly be made for the linear-response calculation at non-zero temperature. Here, the ground state is not shifted by the voltage, but physical properties do not depend so much on the zero-temperature ground state as on the states whose filling fractions are the ones of thermal equilibrium (see [9] for more details on this). Nevertheless, Skorik agrees that the zero-temperature S matrix is the appropriate one to use here, and that it does not acquire any sort of temperature dependence. As discussed in detail previously [1], the only effect of the temperature on the Boltzmann equation is the appearance of the non-trivial filling fractions. To use another language, we are still describing the problem in terms of the "bare" particles (we mean bare in the sense of thermal fluctuations, not quantum fluctuations). The integrability means that these bare particles are a valid basis for the problem and that they still scatter one-by-one off each other and off the impurity despite the macroscopic number of particles excited around a given particle at non-zero temperature. (More precisely, this means that we do calculations in the extremely dilute limit where a particle description is valid; the central assumption of this and all thermodynamic Bethe ansatz computations is that no phase transition interferes with the continuation of the result to the regime of finite densities.)

Consider now a non-zero voltage, and let us discuss the simplest case of $T = 0$. The Fermi sea is, indeed, filled with *bare* (with respect to the zero-voltage sea) quasiparticles, as discussed by Skorik. However, each of these scatters with the same, field-independent S-matrix on the impurity. It is this S matrix that is used to build the asymptotic states, it is therefore the one that appears in the Boltzmann equation in [1]. Of course, one can be interested in the S-matrix for scattering particles excited *on top* of the finite-field sea: by scattering these particles through the impurity and the sea, one gets then a field dependent result. A computation of that type was performed for the the Kondo problem in a magnetic field in [10], as mentioned by Skorik. Such a dressed S matrix has actually appeared in other computations we made, such as low frequency AC properties [8]. But as far as the DC conductance is concerned, the only object necessary is the one to build asymptotic states, and it is the "bare S matrix" as used in [1]. In fact, even if one wanted to use the "dressed" S matrix in our Boltzmann equation, nothing would change. This is because, at the reflection less points of the sine-Gordon model, to which we have restricted, the bulk scattering is diagonal, so the dressing is a mere *phase* (it is also the phase that is discussed in [10].) Since in our Boltzmann-equation approach the DC current depends only on probabilities, i.e. modulus squared of S matrix elements, the V-dependence of this S-matrix would drop out anyway. This, we think, invalidates Skorik's criticism.

There is one major difference between non-zero temperature and non-zero voltage, which might have created some confusion: As emphasized correctly by Skorik, diagonalizing the BSGM directly at nonzero voltage is a difficult exercise, since the boundary interaction does not conserve the charge. It was not done in [1]: there, we argued instead that, physically, the role of the voltage is to fix the populations of quasiparticles; it is applied far from the impurity, and **hence**, since the excitations are localized solitons, in a region where the effect of the impurity is negligible. Thus at least when computing DC

transport properties through the impurity, the effect of the voltage is to shift the Fermi sea but does not interfere with the one-by-one scattering off the impurity (this is actually one of the basic ideas underlying the Landauer-Büttiker approach).

In fact, it is also possible to diagonalize the BSGM with a voltage, and to confirm this physical argument [11,8,7,12]. The main idea is to observe that the physical properties of the BSGM are the same as the properties of another model, where the boundary interaction $\cos\phi$ is replaced by another interaction of the Kondo type $S^+e^{-i\phi}+S^-e^{i\phi}$, where the spin is taken in, either a cyclic representation of the $U_q sl(2)$ algebra [13], or a representation of the oscillator algebra [12]. The key property of these representations is that all monomials (of total vanishing charge) in $S^\pm$ have the same trace. One can then compare the perturbative computations of properties like the partition function, or the conductance using the Keldysh formalism [14] in the two models, and prove their equivalence (this method is used in [15] in the case $g=\frac{1}{2}$, where the $S^\pm$ are equivalent to boundary fermions). The advantage of this new formulation is that there is a conserved charge, the sum of the quasiparticles charge and the spin S^z of the boundary degree of freedom. The voltage is then included by shifting the field $\phi \to \phi + gVt$ in the boundary interaction [3,14,15]. As discussed in the appendix of [7], this is equivalent to not shifting ϕ, but applying a magnetic field on the boundary spin, $h=gV$. Still in the appendix of [7], it is explained how the problem with a field applied only to the boundary spin, and the problem with a field applied to the spin **and** another field applied to the bulk $U(1)$ charge are related by a unitary transformation (this is closely related to the behavior of the electrons-impurity susceptibility in the Kondo problem, as discussed recently in [16]. This transformation shifts the overall charge by a constant, but does not modify difference of $U(1)$ charges). We can then consider the problem where the field is coupled to the conserved charge, which is of course trivial to diagonalize. Asymptotic states can then be written explicitly for this auxiliary problem with impurity **and** voltage; they involve of course the same S matrix as the ones without voltage (the same occurs in the $g=\frac{1}{2}$ case [15]). Scattering eigenstates in the traditional sense can be constructed, and the Landauer-Büttiker approach can then be applied to compute the current, the DC fluctuations [17], and some of the AC properties [8].

In conclusion, the formula proposed in [1] does not suffer from Skorik's criticisms. For the reader who does not want to follow the detailed arguments presented above, we observe that our formula has passed successfully more tests than recognized in Skorik's comment. In addition to the case free-fermion case $g=\frac{1}{2}$, our formula reproduces the correct result for $g \to 0$ and $g=1$. Even though the latter case consists of free fermions in the unfolded version of the problem, **it** is highly non trivial in our folded point of view: indeed, the solitons and antisolitons scatter then with an $SU(2)$ invariant S matrix, identical to the one in the Kondo problem. The former case, while simple to study in the classical limit, is highly non-trivial from the point of view of integrability: it involves taking the complicated limit of an interacting quantum problem, where an infinity of quasiparticles scatter, all with non trivial S matrices, and the filling fractions of the solitons and antisolitons are far from simple. The final result for the $g \to 0$ limit, expressed in terms of Bessel functions of imaginary arguments) [12,18], agrees with the result obtained using a Fokker-Planck equation [19]. In addition, for any g our formula displays the right behavior in the strong and weak backscattering limits, the existence of a maximum for large enough voltage which is expected on physical grounds [1]. Finally, it also obeys a duality (proven at $T=0$ in [1] and related with very natural analyticity conjectures at $T>0$ in [12]) between weak and strong backscattering, where the appropriate tunneling particles are Laughlin quasiparticles and electrons respectively. This duality, while not established rigorously prior to our work, is considered highly desirable on physical grounds [14,15]. We thus see no reason to cast doubts on our result, and hope that sooner or later, it will be compared favorably with numerical simulation or experimental data.

---

[1] P. Fendley, A. Ludwig, H. Saleur, Phys. Rev. Lett. 74 (1995) 3005, cond-mat/9408068; 75 (1995) 2196, cond-mat/9505031; Phys. Rev. B 52 (1995) 8934, cond-mat/9503172.

[2] S. Skorik, cond-mat/9708163

[3] C. Kane, M.P.A. Fisher, Phys. Rev. B 46 (1992) 15523; Phys. Rev. B52 (1995) 17393, cond-mat/9506116.

[4] A.Y. Alekseev, V.V. Cheianov, J. Fröhlich, cond-mat/9607144

[5] R. Landauer, Phys. Rev. B47 (1993) 16427; M. Büttiker, Phys. Rev. B46 (1992) 12485.

[6] P.Fendley, H. Saleur, N. Warner, Nucl. Phys. B 430 (1994) 577, hep-th/9406125

[7] F. Lesage, H. Saleur, Nucl. Phys. B 493 (1997) 613, cond-mat/9612050

[8] F. Lesage, H. Saleur, Nucl. Phys. B 490 (1997) 543, cond-mat/9611025.

[9] A. Leclair, F. Lesage, S. Sachdev, H. Saleur, Nucl.Phys. B482 (1996) 579, cond-mat/9606104

[10] N. Andrei, Phys. Lett. A87 (1982) 299.

[11] P. Fendley, F. Lesage, H. Saleur, J. Stat. Phys. 85 (1996) 211, cond-mat/9510055

[12] V.V. Bazhanov, S. Lukyanov, A. B. Zamolodchikov, hep-th/9604044

[13] P. Fendley, H. Saleur, Phys. Rev. Lett. 75 (1995) 4492, cond-mat/9506104

[14] D. Freed, C. Chamon, X. Wen, Phys. Rev. B 51 (1995) 2363, cond-mat/9408064

[15] D. Freed, C. Chamon, X. Wen, Phys. Rev. B 53 (1996)

4033, cond-mat/9507064
[16] V. Barzykin, I. Affleck, cond-mat/9708039
[17] P. Fendley and H. Saleur, Phys. Rev. B 54 (1996) 10845, cond-mat/9601117
[18] P. Fendley, M.P.A. Fisher and H. Saleur, unpublished.
[19] H.H. Lin and M.P.A. Fisher, Phys. Rev. B54 (1996) 10593, cond-mat/9510156

P. Fendley[1], A.W.W. Ludwig[2] and H. Saleur[3]

[1] Department of Physics, University of Virginia, Charlottesville, VA 22901
[2] Physics Department, University of California, Santa Barbara, CA 93106
[3] Department of Physics, University of Southern California, Los Angeles CA 90089-0484